Higher-spin dynamics and Chern-Simons theories

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ABSTRACT

We review the construction of consistent higher-spin theories based on Chern-Simons actions. To this end we first introduce the required higher-spin algebras and discuss curvature and torsion tensors in an unconstrained way. Finally we perform a perturbative analysis of the Chern-Simons theory in $D = 5$ for a non-maximally symmetric AdS_4 background and obtain the required four-dimensional Frønsdal equations in the compensator formulation.

Presented at the RTN workshop 'ForcesUniverse', Valencia, Spain, October 2007.

April 2008

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1 Introduction

Higher spin (HS) theories have attracted increasing interest not only due to their prominent appearance in string theory, but also as a challenging problem on its own. In fact, until today the problem of finding consistently interacting HS theories (e.g. coupled to gravity) remains without a satisfactory solution. The main obstacle of formulating interactions for, say, massless HS fields is due to the fact that these need to permit a HS gauge symmetry in order to eliminate the longitudinal degrees of freedom. In particular, for fields with spin higher than 3/2 this seems to rule out the possibility of consistent couplings to gravity [1, 2].

One approach to circumvent the no-go theorems for gravity-HS couplings has been pioneered by Vasiliev [3, 4]. It is based on the gauging of certain *infinite-dimensional* HS algebras in a similar spirit as supergravity theories can be viewed as gauge theories of (AdS-)supergroups. However, the actual formulation of the dynamics is a severe problem since any standard coupling like the Einstein-Hilbert term singles out part of the gauge field (as the vielbein) and therefore breaks the symmetry. A manifestly invariant formulation for HS gauge theories is given by the so-called unfolded formulation [5], which is, however, only defined at the level of the equations of motion. In contrast, a fully HS invariant action principle was unknown, with the only exception being the Chern-Simons theory in $D = 3$ constructed in [6].

Here we are going to review [7], in which the construction of Chern-Simons theories based on HS algebras has been extended to generic odd dimensions. While the three-dimensional theory is purely topological, remarkably, this is not so in higher dimensions. To be more precise, around maximally symmetric backgrounds there are still no non-trivial excitations, but around less symmetric solutions there are [8]. We will see explicitly that linearizing the Chern-Simons theory in $D = 5$ around an $AdS_4 \times S^1$ solution, gives precisely rise to the required Frønsdal equations for free HS fields on AdS_4. To this end we will first introduce in sec. 2 a formulation of HS theories in terms of frame-like fields based on HS algebras.

2 Higher-spin algebra and geometry

The HS gauge algebras are infinite-dimensional extensions of the AdS algebra $\mathfrak{so}(D-1, 2)$ in D dimensions. The latter is spanned by M_{AB}, $A, B = 1, \ldots, D + 1$, satisfying the standard algebra

$$[M_{AB}, M_{CD}] = \eta_{BC} M_{AD} - \eta_{AC} M_{BD} - \eta_{BD} M_{AC} + \eta_{AD} M_{BC} , \qquad (2.1)$$

where η_{AB} denotes the $\mathfrak{so}(D - 1, 2)$ invariant metric. The HS algebra $\mathfrak{ho}(D - 1, 2)$ is in turn given by the enveloping algebra $\mathcal{U}(\mathfrak{so}(D - 1, 2))$ of the AdS group, divided by a

certain ideal,

$$\mathfrak{ho}(D-1,2) \;=\; \mathcal{U}(\mathfrak{so}(D-1,2))/\mathcal{I} \,. \tag{2.2}$$

More precisely, the enveloping algebra is spanned by all polynomials in M_{AB}, while modding out some ideal reduces these to a certain subclass corresponding to restricted Young tableaux. A minimal choice then leads to HS generators transforming as tensors in two row Young tableau under $\mathfrak{so}(D-1,2)$:

$$Q_{A(s-1),B(s-1)} : \qquad \qquad , \tag{2.3}$$

where we employed the notation $A(s) = A_1 \cdots A_s$. As we will see below, these give precisely rise to all states which carry integer spin. Other choices would contain mixed HS states corresponding to fields in mixed Young tableaux representations, which do appear in dimensions higher than four and in particular in string theory. That $\mathfrak{ho}(D-1,2)$ is a consistent Lie algebra can be proved, for instance, by means of an explicit realization in terms of vector oscillators. However, the Lie brackets are not known in a closed form, and so we will here focus only on the lowest-order terms, which is sufficient for our linearized analysis below. Explicitly, this means that we focus on the Lie brackets

$$[M_{AB}, Q_{C(s-1),D(s-1)}] \;=\; -4(s-1)\eta_{A\langle C_{s-1}} Q_{|B|C(s-2),D(s-1)\rangle} \,, \tag{2.4}$$

which are fixed by representation theory. Here, brackets $\langle\,\rangle$ denote Young projection according to (2.3).

Let us now examine the gauge theory based on a HS algebra in more detail. In order to read off the 'physical' fields contained in a gauge connection based on $\mathfrak{ho}(D-1,2)$, we need to split the generators (2.3) into Lorentz covariant tensors, i.e. we decompose the generators into $Q_{a(s-1),b(t)}$ for $0 \leq t \leq s-1$ with Lorentz indices $a, b, \ldots = 1, \ldots, D$,

$$Q_{a(s-1),b(t)} : \qquad \qquad .$$

Next we introduce the Lie algebra valued gauge field,

$$\mathcal{A}_\mu \;=\; \bar{e}_\mu{}^a P_a + \frac{1}{2}\bar{\omega}_\mu{}^{ab} M_{ab} + \sum_{s=3}^{\infty} \mathcal{W}_\mu^{(s)} \,, \tag{2.5}$$

where $\bar{e}_\mu{}^a$ and $\bar{\omega}_\mu{}^{ab}$ denote vielbein and spin connection of the background geometry, corresponding to the translation and Lorentz generators, respectively. The spin-s contribution is given by

$$\mathcal{W}_\mu^{(s)} \;=\; \frac{1}{(s-1)!}e_\mu{}^{a(s-1)}Q_{a(s-1)} + \sum_{t=1}^{s-1} \frac{s-t}{s!t!}\omega_\mu{}^{a(s-1),b(t)}Q_{a(s-1),b(t)} \,. \tag{2.6}$$

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The fields $\omega_\mu{}^{a(s-1),b(t)}$ will be interpreted in the following as HS connections, while $e_\mu{}^{a(s-1)}$ corresponds to the physical spin-s field, or the generalized vielbein. The non-abelian curvature derived from (2.4) decomposes into

$$\mathcal{F}_{\mu\nu}^{(s)} = \sum_{t=0}^{s-2} \frac{s-t}{s!t!} T_{\mu\nu}{}^{a(s-1),b(t)} Q_{a(s-1),b(t)} + \frac{1}{s!(s-1)!} R_{\mu\nu}{}^{a(s-1),b(s-1)} Q_{a(s-1),b(s-1)} \;, \quad (2.7)$$

whose explicit form is given by

$$R_{\mu\nu}{}^{a(s-1),b(s-1)} = \bar{D}_\mu \omega_\nu{}^{a(s-1),b(s-1)} + 2(s-1)\Lambda \omega_\mu{}^{\langle a(s-1),b(s-2)} \bar{e}_\nu{}^{b_{s-1}\rangle} - (\mu \leftrightarrow \nu) \;,$$

$$T_{\mu\nu}{}^{a(s-1),b(t)} = \bar{D}_\mu \omega_\nu{}^{a(s-1),b(t)} + \omega_\mu{}^{a(s-1),b(t)c} \bar{e}_{\nu c} \quad\quad (2.8)$$

$$+ t(s-t+1)\Lambda \omega_\mu{}^{\langle a(s-1),b(t-1)} \bar{e}_\nu{}^{b_t\rangle} - (\mu \leftrightarrow \nu) \;,$$

where $\bar{D}_\mu$ denotes the background Lorentz covariant derivative. The components (2.8) will be interpreted as HS Riemann and torsion tensors. Finally we give the non-abelian HS gauge transformations, $\delta \mathcal{A}_\mu = D_\mu \epsilon = \partial_\mu \epsilon + [\mathcal{A}_\mu, \epsilon]$, under which the linearized curvatures (2.8) are invariant:

$$\delta_\epsilon e_\mu{}^{a(s-1)} = \bar{D}_\mu \epsilon^{a(s-1)} - \epsilon^{a(s-1),c} \bar{e}_{\mu c} \;,$$

$$\delta_\epsilon \omega_\mu{}^{a(s-1),b(t)} = \bar{D}_\mu \epsilon^{a(s-1),b(t)} - \epsilon^{a(s-1),b(t)c} \bar{e}_{\mu c} - t(s-t+1)\Lambda \epsilon^{\langle a(s-1),b(t-1)} \bar{e}_\mu{}^{b_t\rangle} \;. \quad (2.9)$$

The symmetry parameterized by $\epsilon^{a(s-1)}$ will give rise to the physical HS symmetry. In addition, the symmetries given by $\epsilon^{a(s-1),b(t)}$ for $t \geq 1$ act as Stückelberg shift symmetries, that generalize the linearized Lorentz transformations of Einstein gravity.

Before we proceed let us make contact with the Frønsdal formulation [9], in which HS fields are represented by totally symmetric tensors $h_{\mu_1\ldots\mu_s}$ of rank s, satisfying the field equations

$$\mathcal{F}_{\mu_1\cdots\mu_s} = \Box h_{\mu_1\cdots\mu_s} - s\nabla_{(\mu_1}\nabla \cdot h_{\mu_2\cdots\mu_s)} + \frac{s(s-1)}{2}\nabla_{(\mu_1}\nabla_{\mu_2} h'_{\mu_3\cdots\mu_s)} + \mathcal{O}(\Lambda) = 0 \;, (2.10)$$

where $\nabla\cdot$ denotes the divergence and h' the trace in the AdS metric. The relation between the HS fields $e_\mu{}^{a(s-1)}$ encountered above and the Frønsdal fields is precisely analogous to the relation between vielbein and metric in general relativity. In the latter, the torsion constraint allows to solve for the spin connection in terms of the vielbein. After gauge-fixing the Lorentz symmetry, only the symmetric part of the vielbein survives, which then coincides with the ordinary metric tensor. In the HS case, the torsion constraints we are going to impose are given by

$$T_{\mu\nu}{}^{a(s-1),b(t)} = 0 \;, \quad\quad 0 \leq t \leq s-2 \;. \quad\quad (2.11)$$

In order to solve this chain of constraints, it turns out to be convenient to gauge-fix the Stückelberg shift symmetries in (2.9). The lowest component of the HS connection then

amounts to the totally symmetric Frønsdal field $h_{\mu_1 \ldots \mu_s} \equiv e_{(\mu_1 | \mu_2 \ldots \mu_s)}$, where we converted all indices into curved ones. The gauge symmetry (2.9) then reads

$$\delta_\epsilon h_{\mu_1 \ldots \mu_s} = \nabla_{(\mu_1} \epsilon_{\mu_2 \ldots \mu_s)} \, , \tag{2.12}$$

which is the HS symmetry that leaves (2.10) invariant. The first connection can be expressed in terms of derivatives of the physical HS field, the second connection in terms of the first connection, etc. In total, this yields a chain of connections, each being expressible in terms of t derivatives of the physical HS field. Their explicit form is given by

$$
\begin{aligned}
\omega_{\mu | \nu(s-1), \rho(t)} &= \frac{s}{s-t} \nabla_{\langle \rho_1} \cdots \nabla_{\rho_t} h_{\nu(s-1) \rangle \mu} \\
&\quad + \sum_{k=1}^{[t/2]} \Lambda^k \gamma_{k,t} g_{\langle \rho_1 \rho_2} \cdots g_{\rho_{2k-1} \rho_{2k}} \nabla_{\rho_{2k+1}} \cdots \nabla_{\rho_t} h_{\nu(s-1) \rangle \mu} \, , \tag{2.13}
\end{aligned}
$$

where $g_{\mu\nu}$ is the AdS metric, and we refer to [10] for the coefficients $\gamma_{k,t}$.

Next we are going to explain in which way the free HS dynamics is encoded in this geometrical formalism. First of all it is puzzling how to obtain sensible second order field equations (the Frønsdal equations (2.10)), since the HS invariant Riemann tensor is an s-derivative object in the physical HS field. However, in [11–13] it has been shown in flat space that the HS Einstein equation — stating vanishing of the HS Ricci tensor — gives effectively rise to second order equations through local integrations, in which the 'integrations constants' correspond to gauge degrees of freedom. To explain this, let us focus on the first non-trivial case, namely a spin-3 field on Minkowski space. Inserting (2.13) into the HS Riemann tensor (2.8) in the limit $\Lambda = 0$ and taking the trace yields [14]

$$R_{\mu\nu \; \rho\sigma,}{}^\lambda{}_\lambda \equiv (\text{Ric})_{\mu\nu | \rho\sigma} = 2 \partial_{[\mu} \mathcal{F}_{\nu] \rho\sigma} = 0 \, . \tag{2.14}$$

Here $\mathcal{F}$ is the Frønsdal operator defined in (2.10). Equation (2.14) shows that the Ricci tensor is a curl, which can therefore be locally integrated by virtue of the Poincaré lemma, resulting in $\mathcal{F}_{\mu\nu\rho} = \partial_\mu \alpha_{\nu\rho}$. Since the right-hand side has to be totally symmetric, this implies $\alpha_{\nu\rho} = \partial_\nu \partial_\rho \alpha$, i.e. in total

$$\mathcal{F}_{\mu\nu\rho} = \partial_\mu \partial_\nu \partial_\rho \alpha \, . \tag{2.15}$$

These are the so-called compensator equations [15, 16]. They possess a larger symmetry than the actual Frønsdal equations $\mathcal{F} = 0$. While the latter are invariant only under so-called constrained transformations with $\epsilon' = \epsilon^\mu{}_\mu = 0$, the compensator equations are completely invariant under (2.12) by virtue of the shift transformation $\delta_\epsilon \alpha = \epsilon'$ on the compensator α. These shift symmetries can in turn be used to set compensator to zero, and so one recovers precisely the spin-3 Frønsdal equations (2.10). In other words, despite being of higher derivative order, (2.14) correctly describes a massless spin-3 field on flat space. It has been shown in [10] that this pattern generalizes to all HS fields on AdS, i.e. the HS Einstein equations as in (2.14) correctly account for the dynamics of massless HS fields on flat space as well as AdS.

3   Higher-spin Chern-Simons theories

In this section we introduce Chern-Simons theories based on HS algebras. To start with, we recall that Chern-Simons actions are gauge-invariant and topological (in the sense that they do not depend on a metric). In $D = 5$, which is the generic situation we will focus on the following, the action for a gauge connection $\mathcal{A}$ is given by

$$S = \int_{M_5} \left\langle \mathcal{A} \wedge d\mathcal{A} \wedge d\mathcal{A} + \frac{3}{2} d\mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A} + \frac{3}{5} \mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A} \right\rangle , \qquad (3.1)$$

where $\langle \, \rangle$ denotes a cubic invariant of the gauge algebra. Denoting the components of this invariant tensor by $g_{\mathcal{ABC}}$, the field equations derived from (3.1) read

$$g_{\mathcal{ABC}} F^{\mathcal{B}} \wedge F^{\mathcal{C}} \;=\; 0 . \qquad (3.2)$$

Despite of its topological origin, this theory is not dynamically trivial. For instance, in [8] it has been shown that specific forms of gravity in odd dimensions (with so-called Lovelock terms) can be viewed as Chern-Simons gauge theories based on the AdS group $SO(D - 1, 2)$ with cubic invariant

$$\langle M_{AB} M_{CD} M_{EF} \rangle \;=\; \varepsilon_{ABCDEF} . \qquad (3.3)$$

While expanding the resulting action around the maximally-symmetric AdS_5 solution, i.e. for $\langle F^{\mathcal{A}} \rangle = 0$, does not give rise to a non-vanishing propagator, this is not so for generic backgrounds with $\langle F^{\mathcal{A}} \rangle \neq 0$. In fact, there is a $AdS_4 \times S^1$ solution, characterized by

$$\bar{R}^{\alpha\beta} + \Lambda \bar{e}^{\alpha} \wedge \bar{e}^{\beta} \;=\; 0 , \qquad \bar{R}^{\alpha 4} + \Lambda \bar{e}^{\alpha} \wedge \bar{e}^{4} \;\neq\; 0 , \qquad (3.4)$$

(with four-dimensional indices $\alpha, \beta, \ldots = 0, \ldots, 3$), which precisely propagates a *four-dimensional* graviton.

Let us now turn to the construction of a Chern-Simons theory based on a HS extension of $\mathfrak{so}(D - 1, 2)$. As these will be gauge-invariant and extend the gravitational theory discussed above, they provide a coupling to gravity that is consistent with the HS gauge symmetries and thus circumvent the no-go theorems. Since the existence of non-trivial infinite-dimensional HS algebras has already been established, the only thing left is to define a cubic invariant of the HS algebra, which extends (3.3). To this end we have to introduce some mathematical machinery, notably the BCH star product. The latter defines an associative product on the enveloping algebra $\mathcal{U}(\mathfrak{so}(D - 1, 2))$ by

$$F(M) \star G(M) \;=\; \exp\left(M_{AB} \Lambda^{AB}(\partial_N, \partial_{N'}) \right) F(N) G(N') \Big|_{N=M, N'=M} , \qquad (3.5)$$

where $F(M) \in \mathcal{U}(\mathfrak{so}(D-1, 2))$, etc., are polynomials in the M_{AB}. Here ∂_N is a short-hand notation for $\partial/\partial N_{AB}$, and $\Lambda^{AB} = -\Lambda^{BA}$ is defined through the Baker-Campbell-Hausdorff relation via

$$\exp Q \exp Q' \;=\; \exp\left(Q + Q' + \Lambda^{AB}(Q, Q') M_{AB} \right) , \qquad (3.6)$$

with $Q = Q^{AB} M_{AB}$ and $Q' = Q'^{AB} M_{AB}$ for some anti-symmetric tensors Q^{AB} and Q'^{AB}. Next we define a sequence of traces by insertion of a differential operator Δ that involves the epsilon tensor,

$$\mathrm{Tr}_k\big(F(M)\big) \;=\; \mathrm{tr}\big(\Delta^k[F(M)]\big) \;\equiv\; \Delta^k[F(M)]\big|_{M=0}\,, \tag{3.7}$$

$$\Delta \;=\; \varepsilon_{A_1 \cdots A_3 B_1 \cdots B_3} \frac{\partial}{\partial M_{A_1 B_1}} \cdots \frac{\partial}{\partial M_{A_3 B_3}}\,. \tag{3.8}$$

It has been proved in [7] that these traces are *cyclic*. We then define a cubic tensor as

$$\big\langle T_s, T_{s'}, T_{s''} \big\rangle \;:=\; \sum_{k=1}^{\infty} \alpha_k \mathrm{Tr}_k\big(\{T_s, T_{s'}\}_\star \star T_{s''}\big)\,, \tag{3.9}$$

where α_k are as yet undetermined coefficients. If we view the enveloping algebra $\mathcal{U}(\mathfrak{so}(D-1,2))$ as a Lie algebra — with the star commutator as bracket —, (3.9) is invariant under its adjoint action. In fact, acting with an arbitrary element of $\mathcal{U}(\mathfrak{so}(D-1,2))$ on (3.9) yields by virtue of the associativity of (3.5) and the cyclicity of (3.7) zero. Moreover, the invariant tensor (3.9) has been defined such that it reduces for the AdS subalgebra precisely to (3.3), which fixes the first coefficient to be $\alpha_1 = \frac{1}{12}$. Thus the resulting HS Chern-Simons theory provides a consistent HS extension of Lovelock gravity in $D=5$.

However, in the enveloping algebra leaving invariant (3.9) we did not yet mod out an ideal $\mathcal{I}$, and so the resulting HS theory contains various kinds of mixed Young-tableau representations. At the same time, we have an infinite number of coefficients α_k to our disposal, which might be fixed by invariance once a more minimal choice of HS algebra has been made. We will leave an exhaustive analysis of the possible ideals in (2.2) and their constraints, if any, on the coefficients α_k for future work.

Up to now we established the existence of HS theories that are consistent in the sense that they exhibit by construction a HS gauge symmetry. It remains to be shown that its dynamical content in the free limit correctly describes massless HS modes. Put differently, we have to show that one recovers the expected free field equations of Frønsdal [9]. As we discussed above, for the purely gravitational part, an expansion around AdS_5 does not give rise to dynamical degrees of freedom. The same holds in presence of HS fields. In fact, linearizing the field equations (3.2) around a given background geometry gives

$$g_{\mathcal{ABC}} \mathcal{R}^{\mathcal{B}}_{\mathrm{AdS}} \wedge R^{\mathcal{C}}_{\mathrm{HS}} \;=\; 0\,, \tag{3.10}$$

where $\mathcal{R}_{\mathrm{AdS}}$ denotes the AdS covariant Riemann tensor of the background geometry and R_{HS} the (linearized) HS contribution. Therefore, in case of an AdS_5 background with $\mathcal{R}_{\mathrm{AdS}} = 0$ the linearized field equations are identically satisfied and to not lead to any (perturbative) dynamics. What we can do instead is to expand around a non-maximally-symmetric solution as the AdS_4 solution in (3.4). Inserting the latter into (3.10) and assuming vanishing HS torsion, yields precisely the HS Einstein equations for the four-dimensional part of the HS fields. As we saw in the previous section, even

though these equations are of higher-derivative order, they are nevertheless equivalent to the (AdS)-Frønsdal equations, or in other words, the higher derivatives correspond to gauge degrees of freedom. Thus the Chern-Simons action (3.1) in $D = 5$ based on a HS algebra describes in particular the dynamics of all four-dimensional massless fields with integer spin. In this sense, it is possible to solve the four-dimensional HS problem in a somewhat holographic way by going to a five-dimensional topological theory!

We close with a few comments on possible impacts and generalizations of these theories. First of all we note that the construction directly extends to all odd dimensions [7]. It has been conjectured already some time ago that M-theory might actually have an interpretation as a Chern-Simons gauge theory in $D = 11$ based on $OSp(1|32)$ [17]. Since M-theory should contain the infinite tower of HS states described by 10-dimensional string theory, it is tempting to speculate that the right framework might be a Chern-Simons theory based on a HS extension of $OSp(1|32)$. The crucial test would be to see whether such a theory permits backgrounds that are 10-dimensional and propagate massive HS modes via some sort of spontaneous symmetry breaking.

Acknowledgments

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